Physics Updates for 2025

Maple provides a state-of-the-art environment for algebraic computations in Physics, with emphasis on ensuring that the computational experience is as natural as possible. The theme of the Physics project for Maple 2025 has been the consolidation of the functionality introduced in previous releases, speed-up of several key internal operations, and significant enhancements regarding functional differentiation, in flat and curved spacetimes. For that purpose a *significant* extension of the algorithms to simplify tensorial expressions in curved spaces was performed, specially for handling expressions involving non covariant derivatives of tensor fields as well as derivatives of Christoffel symbols.

As part of its commitment to providing the best possible computational environment in Physics, Maplesoft launched a Maple Physics: Research and Development website in 2014, which enabled users to download research versions of the package, ask questions, and provide feedback. The results from this accelerated exchange have been incorporated into the Physics package in Maple 2025. The presentation below illustrates both the novelties and the kind of mathematical formulations that can now be performed.

Lagrange Equations and simplification of tensorial expressions in curved spacetimes

LagrangeEquations is a Physics command introduced in 2023 taking advantage of the functional differentiation capabilities of the Physics package. This command can handle tensors and vectors of the Physics package as well as derivatives using vectorial differential operators (see d_ and Nabla), works by performing functional differentiation (see Fundiff), and handles 1^{st}, and higher order derivatives of the coordinates in the Lagrangian automatically. LagrangeEquations receives an expression representing a Lagrangian and returns a sequence of Lagrange equations with as many equations as coordinates are indicated. The number of parameters can also be many. For example, in electrodynamics, the "coordinate" is a tensor field $A_{\mu}(x, y, z, t)$, there are then four coordinates, one for each of the values of the index μ, and there are four parameters (x, y, z, t) .

New in Maple 2025, the "coordinates" can now also be *the components of the metric tensor in a curved spacetime*, in which case the equations returned are Einstein's equations. Also new, instead of a coordinate or set of them, you can pass the keyword *EnergyMomentum*, in which case the output is the conserved energy-momentum tensor of the physical model represented by the given Lagrangian *L*.

Examples

$with(Physics)$:

$Setup(mathematicalnotation = true, coordinates = cartesian)$

$$\textit{Systems of spacetime coordinates are: } \{X = (x, y, z, t)\}$$

$$[coordinatesystems = \{X\}, mathematicalnotation = true] \tag{1}$$

The $\lambda\,\Phi^4$ model in classical field theory and corresponding field equations, as in previous releases

> $CompactDisplay(\Phi(X))$

$$\Phi(x, y, z, t) \textit{ will now be displayed as } \Phi \tag{2}$$

> $L := \frac{1}{2}\, d\_[\mu](\Phi(X))\, d\_[\mu](\Phi(X)) - \frac{m^2}{2}\, \Phi(X)^2 + \frac{\lambda}{4}\, \Phi(X)^4$

$$L := \frac{\partial_\mu(\Phi)\, \partial^\mu(\Phi)}{2} - \frac{m^2\, \Phi^2}{2} + \frac{\lambda\, \Phi^4}{4} \tag{3}$$

Lagrange's equations

> $LagrangeEquations(L, \Phi)$

$$\Phi^3\, \lambda - \Phi\, m^2 - \square(\Phi) = 0 \tag{4}$$

New: The energy-momentum tensor can be computed as the *Lagrange equations* taking the metric as the *coordinate*, not equating to 0 the result, but multiplying the variation of the action $\frac{\delta S}{\delta g^{\mu,\nu}}$ by $\frac{2}{\sqrt{-|g|}}$ (in flat spacetimes $\sqrt{-|g|} = 1$). For that purpose, you can use the *EnergyMomentum* keyword. You can optionally indicate the indices to be used in the output as well as their covariant or contravariant character

> $LagrangeEquations(L, EnergyMomentum[\mu, \nu])$

$$\mathrm{T}_{\mu,\nu} = \left(-\frac{\lambda\, \Phi^4}{4} + \frac{m^2\, \Phi^2}{2} - \frac{\partial^\beta(\Phi)\, \partial_\beta(\Phi)}{2}\right) g_{\mu,\nu} + \partial_\mu(\Phi)\, \partial_\nu(\Phi) \tag{5}$$

To further compute using the above as the definition for $T_{\mu,\nu}$, you can use the Define command

> $Define(\textbf{(5)})$

$$\textit{Defined objects with tensor properties}$$

$$\{\gamma_\mu, \sigma_\mu, \partial_\mu, g_{\mu,\nu}, \mathrm{T}_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}, X_\mu\} \tag{6}$$

After which the system knows about the symmetry properties and the components of $T_{\mu,\nu}$

> $EnergyMomentum[definition]$

$$\mathrm{T}_{\mu,\nu} = \left(-\frac{\lambda\, \Phi^4}{4} + \frac{m^2\, \Phi^2}{2} - \frac{\partial^\beta(\Phi)\, \partial_\beta(\Phi)}{2}\right) g_{\mu,\nu} + \partial_\mu(\Phi)\, \partial_\nu(\Phi) \tag{7}$$

> $Library\text{:-}IsTensorialSymmetric(EnergyMomentum[\mathrm{mu}, \mathrm{nu}])$

$$true \tag{8}$$

> $EnergyMomentum[\,]$

$$\mathrm{T}_{\mu,\nu} = \tag{9}$$

$$\left[\begin{array}{cc} \frac{\lambda\Phi^4}{4} - \frac{m^2\Phi^2}{2} + \frac{\Phi_x^{\,2}}{2} - \frac{\Phi_y^{\,2}}{2} - \frac{\Phi_z^{\,2}}{2} + \frac{\dot{\Phi}^2}{2} & \cdots \\ \Phi_x\Phi_y & \frac{\lambda\Phi^4}{4} - \frac{m^2\Phi^2}{2} - \cdots \\ \Phi_x\Phi_z & \cdots \\ \Phi_x\dot{\Phi} & \cdots \end{array}\right]$$

New: *LagrangeEquations* takes advantage of the extension of Fundiff to compute functional derivatives in curved spacetimes introduced for Maple 2025, and so it also handles the case of a scalar field *in a curved spacetime*. Set for instance an arbitrary metric

> $g\_[arb]$

Setting lowercaselatin_is letters to represent space indices

The arbitrary metric in coordinates $[x, y, z, t]$

Signature: $(- - - +)$

$$g_{\mu,\nu} = \left[\begin{array}{cccc} f_1(X) & f_2(X) & f_3(X) & f_4(X) \\ f_2(X) & f_5(X) & f_6(X) & f_7(X) \\ f_3(X) & f_6(X) & f_8(X) & f_9(X) \\ f_4(X) & f_7(X) & f_9(X) & f_{10}(X) \end{array}\right] \tag{10}$$

For the action to be a true scalar in spacetime, the Lagrangian density now needs to be multiplied by the square root of the determinant of the metric

> $L := \mathrm{sqrt}(-\%g\_[determinant])\, L$

$$L := \sqrt{-|\mathfrak{g}|}\left(\frac{\partial_\mu(\Phi)\,\partial^\mu(\Phi)}{2} - \frac{m^2\Phi^2}{2} + \frac{\lambda\Phi^4}{4}\right) \tag{11}$$

New: With the extension of the tensorial simplification algorithms for curved spacetimes, the Lagrange equations can be computed arriving directly to the compact form

> *LagrangeEquations*(L, Φ)

$$\Phi^3 \lambda - \Phi m^2 - \nabla_\kappa\left(\partial^\kappa(\Phi)\right) = 0 \tag{12}$$

Comparing with the result **(4)** for the same Lagrangian in a flat spacetime, we see the only difference is that the dAlembertian is now expressed in terms of covariant derivatives D_.

The EnergyMomentum tensor is computed in the same way as when the spacetime is flat

> *LagrangeEquations*$(L, EnergyMomentum[\mu, \nu])$

$$T_{\mu,\nu} = \left(-\frac{\lambda \Phi^4}{4} + \frac{m^2 \Phi^2}{2} - \frac{\partial^\beta(\Phi)\,\partial_\beta(\Phi)}{2}\right) g_{\mu,\nu} + \partial_\mu(\Phi)\,\partial_\nu(\Phi) \tag{13}$$

General Relativity

New: the most significant development in *LagrangeEquations* is regarding General Relativity. It can now compute Einstein's equations directly from the Lagrangian, not using tabulated cases, and properly handling several (traditional or not) alternative ways of presenting the Lagrangian.

Einstein's equations concern the case of a curved spacetime with metric $g_{\mu,\nu}$ as, for instance, the general case of an arbitrary metric set lines above. In the Lagrangian formulation, the *coordinates* of the problem are the components of the metric $g_{\mu,\nu}$, and as in the case of electrodynamics the *parameters* are the spacetime coordinates X^α. The simplest case is that of Einstein's equation in vacuum, for which the Lagrangian density is expressed in terms of the trace of the Ricci tensor by

> $L :=$ sqrt$(-\%g_[determinant])$ *Ricci*$[\alpha, \sim alpha]$

$$L := \sqrt{-|\mathrm{g}|}\, R_\alpha^{\ \alpha} \tag{14}$$

Einstein's equations in vacuum:

> *LagrangeEquations*$(L, g\_[\mu, \nu])$

$$-\frac{g_{\mu,\nu} R_\alpha^{\ \alpha}}{2} + R_{\mu,\nu} = 0 \tag{15}$$

where in the above instead of passing g as second argument, we passed $g_{\mu,\nu}$ to get the equations using those free indices. The tensorial equation computed is also the definition of the Einstein tensor

> *Einstein*$[definition]$

$$G_{\mu,\nu} = -\frac{g_{\mu,\nu} R_\alpha^{\ \alpha}}{2} + R_{\mu,\nu} \tag{16}$$

The Lagrangian L used to compute Einstein's equations **(15)** contains first and second derivatives of the metric. To see that, rewrite L in terms of Christoffel symbols

> $L_C := convert(L, Christoffel)$

$$L_C := \sqrt{-|\mathrm{g}|}\, g^{\alpha,\lambda} \left(\partial_\nu\left(\Gamma^{\nu}{}_{\alpha,\lambda}\right) - \partial_\lambda\left(\Gamma^{\nu}{}_{\alpha,\nu}\right) + \Gamma^{\beta}{}_{\alpha,\lambda}\Gamma^{\nu}{}_{\beta,\nu} - \Gamma^{\beta}{}_{\alpha,\nu}\Gamma^{\nu}{}_{\beta,\lambda}\right) \tag{17}$$

Recalling the definition

> $Christoffel[definition]$

$$\Gamma_{\alpha,\mu,\nu} = \frac{\partial_\nu\left(g_{\alpha,\mu}\right)}{2} + \frac{\partial_\mu\left(g_{\alpha,\nu}\right)}{2} - \frac{\partial_\alpha\left(g_{\mu,\nu}\right)}{2} \tag{18}$$

in L_C the two terms containing derivatives of Christoffel symbols contain second order derivatives of $g_{\mu,\nu}$. Now, it is always possible to add a total spacetime derivative to L_C without changing Einstein's equations (assuming the variation of the metric in the corresponding boundary integrals vanishes), and in that way, in this particular case of L_C, obtain a Lagrangian involving only 1st order derivatives. The total derivative, expressed using the inert ∂ command to see it before the differentiation operation is performed, is

> $TD := \%d\_[\alpha]\left(g\_[\sim mu, \sim nu]\,\mathrm{sqrt}(-\%g\_[determinant])\,\left(-Christoffel[\sim alpha, \mu, \nu]\right.\right.$
> $\left.\left. + g\_[\sim alpha, \mu]\, Christoffel[\sim beta, \nu, \beta]\right)\right)$

$$TD := \partial_\alpha\left(g^{\mu,\nu}\sqrt{-|\mathrm{g}|}\left(\delta_\mu{}^\alpha\Gamma^{\beta}{}_{\beta,\nu} - \Gamma^{\alpha}{}_{\mu,\nu}\right)\right) \tag{19}$$

Adding this term to L_C, performing the ∂ differentiation operation and simplifying we get

> $L_1 := L_C + TD$

$$L_1 := \sqrt{-|\mathrm{g}|}\, g^{\alpha,\lambda} \left(\partial_\nu\left(\Gamma^{\nu}{}_{\alpha,\lambda}\right) - \partial_\lambda\left(\Gamma^{\nu}{}_{\alpha,\nu}\right) + \Gamma^{\beta}{}_{\alpha,\lambda}\Gamma^{\nu}{}_{\beta,\nu} - \Gamma^{\beta}{}_{\alpha,\nu}\Gamma^{\nu}{}_{\beta,\lambda}\right) + \partial_\alpha\left(g^{\mu,\nu}\sqrt{-|\mathrm{g}|}\left(\delta_\mu{}^\alpha\Gamma^{\beta}{}_{\beta,\nu} - \Gamma^{\alpha}{}_{\mu,\nu}\right)\right) \tag{20}$$

> $L_1 := eval\left(L_1, \%d\_ = d\_\right)$

$$L_1 := \sqrt{-|\mathrm{g}|}\, g^{\alpha,\lambda} \left(\partial_\nu\left(\Gamma^{\nu}{}_{\alpha,\lambda}\right) - \partial_\lambda\left(\Gamma^{\nu}{}_{\alpha,\nu}\right) + \Gamma^{\beta}{}_{\alpha,\lambda}\Gamma^{\nu}{}_{\beta,\nu} - \Gamma^{\beta}{}_{\alpha,\nu}\Gamma^{\nu}{}_{\beta,\lambda}\right) + \partial_\alpha\left(g^{\mu,\nu}\right)\sqrt{-|\mathrm{g}|}\left(\delta_\mu{}^\alpha\Gamma^{\beta}{}_{\beta,\nu} - \Gamma^{\alpha}{}_{\mu,\nu}\right) - \frac{g^{\mu,\nu}\left(\delta_\mu{}^\alpha\Gamma^{\beta}{}_{\beta,\nu} - \Gamma^{\alpha}{}_{\mu,\nu}\right)|\mathrm{g}|\, g^{\kappa,\lambda}\,\partial_\alpha\left(g_{\kappa,\lambda}\right)}{2\sqrt{-|\mathrm{g}|}} + g^{\mu,\nu}\sqrt{-|\mathrm{g}|}\left(\delta_\mu{}^\alpha\,\partial_\alpha\left(\Gamma^{\beta}{}_{\beta,\nu}\right) - \partial_\alpha\left(\Gamma^{\alpha}{}_{\mu,\nu}\right)\right) \tag{21}$$

> $L_1 := Simplify\left(L_1\right)$

$$L_1 := \left(\Gamma_{\alpha,\beta,\kappa}\Gamma^{\beta,\alpha,\kappa} - \Gamma_{\alpha,\beta}{}^{\alpha}\Gamma^{\beta\ \kappa}{}_{\kappa}\right)\sqrt{-|\mathrm{g}|} \tag{22}$$

which is a Lagrangian depending *only on 1st order derivatives of the metric* through Christoffel symbols. As expected, the *equations of motion* resulting from this Lagrangian are the same Einstein equations computed in **(15)**

> $LagrangeEquations\left(L_1, g\_[\mu, \nu]\right)$

$$-\frac{R_{\iota}^{\ \iota} g_{\mu,\nu}}{2} + R_{\mu,\nu} = 0 \tag{23}$$

To illustrate the new Maple 2025 tensorial simplification capabilities note that $L_1 \equiv$ **(22)** is no just $L_C \equiv$ **(17)** after discarding its two terms involving derivatives of Christoffel symbols. To verify this, split L_C into the terms containing or not derivatives of Christoffel

> $L_{22}, L_{11} := selectremove\left(has, expand\left(L_C\right), d\_\right)$

$$L_{22}, L_{11} := \sqrt{-|\,\mathrm{g}\,|}\, g^{\alpha,\lambda}\, \partial_{\nu}\left(\Gamma^{\nu}{}_{\alpha,\lambda}\right) - \sqrt{-|\,\mathrm{g}\,|}\, g^{\alpha,\lambda}\, \partial_{\lambda}\left(\Gamma^{\nu}{}_{\alpha,\nu}\right), \sqrt{-|\,\mathrm{g}\,|}\, g^{\alpha,\lambda}\, \Gamma^{\beta}{}_{\alpha,\lambda} \Gamma^{\nu}{}_{\beta,\nu} - \sqrt{-|\,\mathrm{g}\,|}\, g^{\alpha,\lambda}\, \Gamma^{\beta}{}_{\alpha,\nu} \Gamma^{\nu}{}_{\beta,\lambda} \tag{24}$$

Comparing, the total derivative $TD \equiv$ **(19)** is not just $-L_{22}$, but

> $TD = -L_{22} - 2\, L_{11}$

$$\partial_{\alpha}\left(g^{\mu,\nu} \sqrt{-|\,\mathrm{g}\,|}\, \left(\delta_{\mu}^{\ \alpha}\, \Gamma^{\beta}{}_{\beta,\nu} - \Gamma^{\alpha}{}_{\mu,\nu}\right)\right) = -\sqrt{-|\,\mathrm{g}\,|}\, g^{\alpha,\lambda}\, \partial_{\nu}\left(\Gamma^{\nu}{}_{\alpha,\lambda}\right) + \sqrt{-|\,\mathrm{g}\,|}\, g^{\alpha,\lambda}\, \partial_{\lambda}\left(\Gamma^{\nu}{}_{\alpha,\nu}\right) - 2\sqrt{-|\,\mathrm{g}\,|}\, g^{\alpha,\lambda}\, \Gamma^{\beta}{}_{\alpha,\lambda} \Gamma^{\nu}{}_{\beta,\nu} + 2\sqrt{-|\,\mathrm{g}\,|}\, g^{\alpha,\lambda}\, \Gamma^{\beta}{}_{\alpha,\nu} \Gamma^{\nu}{}_{\beta,\lambda} \tag{25}$$

Things like these, $TD = -L_{22} - 2\, L_{11}$, can now be verified directly with the new tensorial simplification capabilities: take the left-hand side minus the right-hand side, evaluate the inert derivative ∂ and simplify to see the equality is true

> $(lhs - rhs)\,($**(25)**$)$

$$\partial_{\alpha}\left(g^{\mu,\nu} \sqrt{-|\,\mathrm{g}\,|}\, \left(\delta_{\mu}^{\ \alpha}\, \Gamma^{\beta}{}_{\beta,\nu} - \Gamma^{\alpha}{}_{\mu,\nu}\right)\right) + \sqrt{-|\,\mathrm{g}\,|}\, g^{\alpha,\lambda}\, \partial_{\nu}\left(\Gamma^{\nu}{}_{\alpha,\lambda}\right) - \sqrt{-|\,\mathrm{g}\,|}\, g^{\alpha,\lambda}\, \partial_{\lambda}\left(\Gamma^{\nu}{}_{\alpha,\nu}\right) + 2\sqrt{-|\,\mathrm{g}\,|}\, g^{\alpha,\lambda}\, \Gamma^{\beta}{}_{\alpha,\lambda} \Gamma^{\nu}{}_{\beta,\nu} - 2\sqrt{-|\,\mathrm{g}\,|}\, g^{\alpha,\lambda}\, \Gamma^{\beta}{}_{\alpha,\nu} \Gamma^{\nu}{}_{\beta,\lambda} \tag{26}$$

> $eval($**(26)**$, \%d\_ = d\_)$

$$\partial_{\alpha}\left(g^{\mu,\nu}\right) \sqrt{-|\,\mathrm{g}\,|}\, \left(\delta_{\mu}^{\ \alpha}\, \Gamma^{\beta}{}_{\beta,\nu} - \Gamma^{\alpha}{}_{\mu,\nu}\right) - \frac{g^{\mu,\nu} \left(\delta_{\mu}^{\ \alpha}\, \Gamma^{\beta}{}_{\beta,\nu} - \Gamma^{\alpha}{}_{\mu,\nu}\right) |\,\mathrm{g}\,|\, g^{\kappa,\lambda}\, \partial_{\alpha}\left(g_{\kappa,\lambda}\right)}{2\sqrt{-|\,\mathrm{g}\,|}} + g^{\mu,\nu} \sqrt{-|\,\mathrm{g}\,|}\, \left(\delta_{\mu}^{\ \alpha}\, \partial_{\alpha}\left(\Gamma^{\beta}{}_{\beta,\nu}\right) - \partial_{\alpha}\left(\Gamma^{\alpha}{}_{\mu,\nu}\right)\right) + \sqrt{-|\,\mathrm{g}\,|}\, g^{\alpha,\lambda}\, \partial_{\nu}\left(\Gamma^{\nu}{}_{\alpha,\lambda}\right) - \sqrt{-|\,\mathrm{g}\,|}\, g^{\alpha,\lambda}\, \partial_{\lambda}\left(\Gamma^{\nu}{}_{\alpha,\nu}\right) + 2\sqrt{-|\,\mathrm{g}\,|}\, g^{\alpha,\lambda}\, \Gamma^{\beta}{}_{\alpha,\lambda} \Gamma^{\nu}{}_{\beta,\nu} - 2\sqrt{-|\,\mathrm{g}\,|}\, g^{\alpha,\lambda}\, \Gamma^{\beta}{}_{\alpha,\nu} \Gamma^{\nu}{}_{\beta,\lambda} \tag{27}$$

> $Simplify($**(27)**$)$

$$0 \tag{28}$$

That said, it is also true that $TD = -L_{22} - 2\,L_{11}$ results in the Lagrangian $L_1 = -L_{11}$, and since the equations of movement don't depend on the sign of the Lagrangian, for this Lagrangian $L_C \equiv$ **(17)** adding the term TD happens to be equivalent to just discarding the terms of L_C involving derivatives of Christoffel symbols.

Also **new** in Maple 2025, due to the extension of Fundiff to compute in curved spacetimes, it is now also possible to compute Einstein's equations from first principles by constructing the action,

> $S := Intc(L, X)$

$$S := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{-|\,\mathrm{g}\,|}\; R_{\alpha}{}^{\alpha}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}t \qquad \textbf{(29)}$$

and equating to zero the functional derivative with respect to the metric. To avoid displaying the resulting large expression, end the input line with ":"

> $EE_{unsimplified} := Fundiff\big(S, g\_[\alpha, \beta]\big) = 0 :$

Simplifying this result, we get an expression in terms of Christoffel symbols and its derivatives

> $EEC := Simplify\big(EE_{unsimplified}\big)$

$$EEC := \frac{\left(2\,\Gamma_{\chi,\iota,\kappa}\Gamma^{\iota,\chi,\kappa} - 2\,\Gamma_{\chi,\iota}{}^{\chi}\,\Gamma^{\iota}{}_{\kappa}{}^{\kappa} - 2\,\nabla_{\iota}\!\left(\Gamma_{\chi}{}^{\chi,\iota}\right) + 2\,\nabla_{\chi}\!\left(\Gamma^{\chi}{}_{\iota}{}^{\iota}\right)\right) g^{\alpha,\beta}}{4}$$
$$+ \frac{\left(\Gamma^{\alpha}{}_{\chi}{}^{\beta} + \Gamma^{\beta}{}_{\chi}{}^{\alpha}\right)\Gamma^{\chi}{}_{\iota}{}^{\iota}}{4} - \frac{\Gamma^{\beta}{}_{\chi,\iota}\,\Gamma^{\chi,\alpha,\iota}}{4} - \frac{\Gamma_{\chi,\iota}{}^{\alpha}\,\Gamma^{\iota,\beta,\chi}}{2} + \frac{\Gamma_{\chi}{}^{\alpha,\beta}\,\Gamma_{\iota}{}^{\chi,\iota}}{2}$$
$$- \frac{\Gamma^{\alpha}{}_{\chi,\iota}\,\Gamma^{\chi,\beta,\iota}}{4} + \frac{\nabla_{\chi}\!\left(\Gamma^{\alpha,\beta,\chi}\right)}{4} + \frac{\nabla_{\chi}\!\left(\Gamma^{\beta,\alpha,\chi}\right)}{4} - \frac{\nabla_{\chi}\!\left(\Gamma^{\chi,\alpha,\beta}\right)}{2} - \frac{\nabla^{\alpha}\!\left(\Gamma^{\beta}{}_{\chi}{}^{\chi}\right)}{4}$$
$$+ \frac{\nabla^{\beta}\!\left(\Gamma_{\chi}{}^{\alpha,\chi}\right)}{2} - \frac{\nabla^{\beta}\!\left(\Gamma^{\alpha}{}_{\chi}{}^{\chi}\right)}{4} = 0 \qquad \textbf{(30)}$$

In this result, we see ∇ *derivatives* of Christoffel symbols, expressed using the D_ command for covariant differentiation. Although, such objects have not the geometrical meaning of a covariant derivative, computationally, they here represent what would be a covariant derivative if the Christoffel symbols were a tensor. For example,

> $\nabla_{\chi}\!\left(\Gamma^{\alpha,\beta,\chi}\right) :$

> $\% = expand(\%)$

$$\nabla_{\chi}\!\left(\Gamma^{\alpha,\beta,\chi}\right) = \partial_{\chi}\!\left(\Gamma^{\alpha,\beta,\chi}\right) + \Gamma^{\alpha}{}_{\chi,\mu}\,\Gamma^{\mu,\beta,\chi} + \Gamma^{\beta}{}_{\chi,\mu}\,\Gamma^{\alpha,\chi,\mu} + \Gamma^{\chi}{}_{\chi,\mu}\,\Gamma^{\alpha,\beta,\mu} \qquad \textbf{(31)}$$

With this computational meaning for the ∇ derivatives of Christoffel symbols appearing in **(30)**, rewrite $EEC \equiv$ **(30)** in terms of the Ricci and Riemann tensors. For that, consider the definition

> $Ricci[definition]$

$$R_{\mu,\nu} = \partial_\alpha\left(\Gamma^{\alpha}{}_{\mu,\nu}\right) - \partial_\nu\left(\Gamma^{\alpha}{}_{\mu,\alpha}\right) + \Gamma^{\beta}{}_{\mu,\nu}\Gamma^{\alpha}{}_{\beta,\alpha} - \Gamma^{\beta}{}_{\mu,\alpha}\Gamma^{\alpha}{}_{\nu,\beta} \tag{32}$$

Rewrite the noncovariant derivatives ∂ in terms of ∇ derivatives using the computational representation **(31)**, simplify and isolate one of them

> *convert*(**(32)**, *D_*)

$$R_{\mu,\nu} = \nabla_\alpha\left(\Gamma^{\alpha}{}_{\mu,\nu}\right) - \Gamma^{\alpha}{}_{\alpha,\kappa}\Gamma^{\kappa}{}_{\mu,\nu} + \Gamma^{\kappa}{}_{\alpha,\mu}\Gamma^{\alpha}{}_{\kappa,\nu} + \Gamma^{\kappa}{}_{\alpha,\nu}\Gamma^{\alpha}{}_{\mu,\kappa} - \nabla_\nu\left(\Gamma^{\alpha}{}_{\alpha,\mu}\right) - \Gamma^{\lambda}{}_{\mu,\nu}\Gamma^{\alpha}{}_{\alpha,\lambda} + \Gamma^{\beta}{}_{\mu,\nu}\Gamma^{\alpha}{}_{\alpha,\beta} - \Gamma^{\beta}{}_{\alpha,\mu}\Gamma^{\alpha}{}_{\beta,\nu} \tag{33}$$

> *Simplify*(**(33)**)

$$R_{\mu,\nu} = \Gamma_{\alpha,\beta,\mu}\Gamma^{\beta\ \alpha}{}_{\nu} - \Gamma_{\beta,\mu,\nu}\Gamma_{\alpha}{}^{\alpha,\beta} + \nabla_\alpha\left(\Gamma^{\alpha}{}_{\mu,\nu}\right) - \nabla_\nu\left(\Gamma_{\alpha,\mu}{}^{\alpha}\right) \tag{34}$$

> *C_to_Ricci* := *isolate*(**(34)**, *D_*[α](*Christoffel*[~*alpha*, μ, ν]))

$$C\_to\_Ricci := \nabla_\alpha\left(\Gamma^{\alpha}{}_{\mu,\nu}\right) = -\Gamma_{\alpha,\beta,\mu}\Gamma^{\beta\ \alpha}{}_{\nu} + \Gamma_{\beta,\mu,\nu}\Gamma_{\alpha}{}^{\alpha,\beta} + R_{\mu,\nu} + \nabla_\nu\left(\Gamma_{\alpha,\mu}{}^{\alpha}\right) \tag{35}$$

Analogously, derive an expression to rewrite ∇ derivatives of Christoffel symbols using the Riemann tensor

> *Riemann*[~*alpha*, β, μ, ν, *definition*]

$$R^{\alpha}{}_{\beta,\mu,\nu} = \partial_\mu\left(\Gamma^{\alpha}{}_{\beta,\nu}\right) - \partial_\nu\left(\Gamma^{\alpha}{}_{\beta,\mu}\right) + \Gamma^{\alpha}{}_{\upsilon,\mu}\Gamma^{\upsilon}{}_{\beta,\nu} - \Gamma^{\alpha}{}_{\upsilon,\nu}\Gamma^{\upsilon}{}_{\beta,\mu} \tag{36}$$

> *convert*(**(36)**, *D_*)

$$R^{\alpha}{}_{\beta,\mu,\nu} = \nabla_\mu\left(\Gamma^{\alpha}{}_{\beta,\nu}\right) + \Gamma^{\kappa}{}_{\mu,\nu}\Gamma^{\alpha}{}_{\beta,\kappa} - \Gamma^{\alpha}{}_{\kappa,\mu}\Gamma^{\kappa}{}_{\beta,\nu} + \Gamma^{\kappa}{}_{\beta,\mu}\Gamma^{\alpha}{}_{\kappa,\nu} - \nabla_\nu\left(\Gamma^{\alpha}{}_{\beta,\mu}\right) - \Gamma^{\lambda}{}_{\mu,\nu}\Gamma^{\alpha}{}_{\beta,\lambda} - \Gamma^{\lambda}{}_{\beta,\nu}\Gamma^{\alpha}{}_{\lambda,\mu} + \Gamma^{\alpha}{}_{\lambda,\nu}\Gamma^{\lambda}{}_{\beta,\mu} + \Gamma^{\alpha}{}_{\mu,\upsilon}\Gamma^{\upsilon}{}_{\beta,\nu} - \Gamma^{\alpha}{}_{\nu,\upsilon}\Gamma^{\upsilon}{}_{\beta,\mu} \tag{37}$$

> *Simplify*(**(37)**)

$$R^{\alpha}{}_{\beta,\mu,\nu} = -\Gamma^{\alpha}{}_{\kappa,\mu}\Gamma^{\kappa}{}_{\beta,\nu} + \Gamma^{\kappa}{}_{\beta,\mu}\Gamma^{\alpha}{}_{\kappa,\nu} + \nabla_\mu\left(\Gamma^{\alpha}{}_{\beta,\nu}\right) - \nabla_\nu\left(\Gamma^{\alpha}{}_{\beta,\mu}\right) \tag{38}$$

> *C_to_Riemann* := *isolate*(**(38)**, *D_*[μ](*Christoffel*[~*alpha*, β, ν]))

$$C\_to\_Riemann := \nabla_\mu\left(\Gamma^{\alpha}{}_{\beta,\nu}\right) = \Gamma^{\alpha}{}_{\kappa,\mu}\Gamma^{\kappa}{}_{\beta,\nu} - \Gamma^{\kappa}{}_{\beta,\mu}\Gamma^{\alpha}{}_{\kappa,\nu} + R^{\alpha}{}_{\beta,\mu,\nu} + \nabla_\nu\left(\Gamma^{\alpha}{}_{\beta,\mu}\right) \tag{39}$$

Substitute these two equations, in sequence, into Einstein's equations *EEC*≡**(30)**

> *Substitute*(*C_to_Riemann*, *C_to_Ricci*, *EEC*)

$$-\frac{\Gamma^{\beta\ \alpha}{}_{\psi}\Gamma^{\psi\ \omega}{}_{\omega}}{4} + \frac{\Gamma^{\beta\ \omega}{}_{\psi}\Gamma^{\psi\ \alpha}{}_{\omega}}{4} + \frac{\Gamma^{\alpha}{}_{\lambda,\mu}\Gamma^{\lambda,\beta,\mu}}{4} - \frac{\Gamma^{\alpha\ \mu}{}_{\lambda}\Gamma^{\lambda\ \beta}{}_{\mu}}{4} + \frac{\Gamma^{\beta}{}_{\nu,\sigma}\Gamma^{\sigma,\alpha,\nu}}{4} - \frac{\Gamma^{\beta\ \nu}{}_{\sigma}\Gamma^{\sigma\ \alpha}{}_{\nu}}{4} + \frac{\Gamma_{o,\zeta}{}^{\beta}\Gamma^{\zeta\,\alpha,o}}{2} - \frac{\Gamma_{o,\zeta}{}^{o}\Gamma^{\zeta\,\alpha,\beta}}{2} - R^{\alpha,\beta} - \frac{\Gamma^{\tau,\alpha,\beta}\Gamma^{\upsilon}{}_{\tau,\upsilon}}{2} \tag{40}$$

$$-\frac{\Gamma_{\chi,\iota}{}^{\alpha}\,\Gamma^{\iota,\beta,\chi}}{2}+\frac{\left(\Gamma^{\alpha}{}_{\chi}{}^{\beta}+\Gamma^{\beta}{}_{\chi}{}^{\alpha}\right)\Gamma^{\chi}{}_{\iota}{}^{\iota}}{4}+\frac{\Gamma^{\upsilon}{}_{\tau}{}^{\beta}\,\Gamma^{\tau}{}_{\upsilon}{}^{\alpha}}{2}-\frac{\Gamma^{\beta}{}_{\chi,\iota}\,\Gamma^{\chi,\alpha,\iota}}{4}+\frac{\Gamma_{\chi}{}^{\alpha,\beta}\,\Gamma^{\chi}{}_{\iota}{}^{\iota}}{2}$$

$$-\frac{\Gamma^{\alpha}{}_{\chi,\iota}\,\Gamma^{\chi,\beta,\iota}}{4}-\frac{\nabla^{\rho1}\left(\Gamma^{\alpha}{}_{\rho1}{}^{\beta}\right)}{4}+\frac{\nabla^{\mu}\left(\Gamma^{\alpha}{}_{\mu}{}^{\beta}\right)}{4}+\frac{\nabla^{\nu}\left(\Gamma^{\beta}{}_{\nu}{}^{\alpha}\right)}{4}-\frac{\nabla^{\beta}\left(\Gamma^{\upsilon}{}_{\upsilon}{}^{\alpha}\right)}{2}$$

$$-\frac{\nabla^{\omega}\left(\Gamma^{\beta}{}_{\omega}{}^{\alpha}\right)}{4}+\frac{\nabla^{\beta}\left(\Gamma_{\alpha9}{}^{\alpha,\alpha9}\right)}{2}-\frac{\Gamma^{\alpha}{}_{\rho}{}^{\beta}\,\Gamma^{\rho}{}_{\rho1}{}^{\rho1}}{4}+\frac{\Gamma^{\alpha}{}_{\rho}{}^{\rho1}\,\Gamma^{\rho}{}_{\rho1}{}^{\beta}}{4}$$

$$+\frac{\Gamma_{\alpha10}{}^{\alpha,\beta}\,\Gamma_{\alpha9}{}^{\alpha10,\alpha9}}{2}-\frac{\Gamma_{\alpha9,\alpha10}{}^{\alpha}\,\Gamma^{\alpha10,\alpha9,\beta}}{2}+\frac{1}{4}\Big(\Big(2\,\Gamma_{\chi,\iota,\kappa}\,\Gamma^{\iota,\chi,\kappa}-2\,\Gamma_{\chi,\iota}{}^{\chi}\,\Gamma^{\iota}{}_{\kappa}{}^{\kappa}$$

$$-2\,\Gamma_{\alpha4,\alpha5,\alpha6}\,\Gamma^{\alpha6,\alpha4,\alpha5}+2\,\Gamma_{\alpha4,\alpha6}{}^{\alpha5}\,\Gamma^{\alpha6}{}_{\alpha5}{}^{\alpha4}-2\,\nabla^{\alpha5}\left(\Gamma_{\alpha4,\alpha5}{}^{\alpha4}\right)+2\,\Gamma^{\alpha1}{}_{\alpha1,\alpha3}\,\Gamma^{\alpha3}{}_{\alpha2}{}^{\alpha2}$$

$$-2\,\Gamma^{\alpha1}{}_{\alpha3}{}^{\alpha2}\,\Gamma^{\alpha3}{}_{\alpha1,\alpha2}+2\,R_{\alpha2}{}^{\alpha2}+2\,\nabla^{\alpha2}\left(\Gamma^{\alpha1}{}_{\alpha1,\alpha2}\right)\Big)\Big)\,g^{\alpha,\beta}\Big)=0$$

Simplify to arrive at the traditional compact form of Einstein's equations

> *Simplify*(**(40)**)

$$\frac{R^{\chi}{}_{\chi}\,g^{\alpha,\beta}}{2}-R^{\alpha,\beta}=0 \qquad \textbf{(41)}$$

Linearized Gravity

Generally speaking, linearizing gravity is about discarding in Einstein's field equations the terms that are quadratic in the metric and its derivatives, an approximation valid when the gravitational field is weak (the deviation from a flat Minkowski spacetime is small). Linearizing gravity is used, e.g. in the study of gravitational waves. In the context of Maple's Physics, the formulation of linearized gravity can be done using the general relativity tensors that come predefined in Physics plus a new in Maple 2025 *Physics:-Library:-Linearize* command.

In what follows it is shown how to linearize the Ricci tensor and through it Einstein's equations. To compare results, see for instance the Wikipedia page for Linearized gravity. Start setting coordinates, you could use *Cartesian, spherical*, *cylindrical*, or define your own.

> *restart*;
with(*Physics*) :
Setup(*coordinates* = *cartesian*)

Systems of spacetime coordinates are: $\{X=(x, y, z, t)\}$

$$[\,coordinatesystems=\{X\}\,] \qquad \textbf{(42)}$$

The default metric when Physics is loaded is the *Minkowski* metric, representing a flat (no curvature) spacetime

> $g\_[\,]$

$$g_{\mu,\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \textbf{(43)}$$

The weakly perturbed metric

Suppose you want to define a small perturbation around this metric. For that purpose, define a perturbation tensor $h_{\mu,\nu}$, that in the general case depends on the coordinates and is not diagonal, the only requirement is that it is *symmetric* (to have it *diagonal*, change *symmetric* by *diagonal*; to have it *constant*, change $\delta_{i,j}(X)$ by $\delta_{i,j}$)

> $\mathrm{h}[\mathrm{mu},\mathrm{nu}] = Matrix\left(4, (i,j) \rightarrow \mathrm{delta}[i,j](X), shape = symmetric\right)$

$$h_{\mu,\nu} = \begin{bmatrix} \delta_{1,1}(X) & \delta_{1,2}(X) & \delta_{1,3}(X) & \delta_{1,4}(X) \\ \delta_{1,2}(X) & \delta_{2,2}(X) & \delta_{2,3}(X) & \delta_{2,4}(X) \\ \delta_{1,3}(X) & \delta_{2,3}(X) & \delta_{3,3}(X) & \delta_{3,4}(X) \\ \delta_{1,4}(X) & \delta_{2,4}(X) & \delta_{3,4}(X) & \delta_{4,4}(X) \end{bmatrix} \quad \textbf{(44)}$$

In the above it is understood that $\delta_{i,j} \ll 1$, so that quadratic or higher powers of it or its derivatives can be approximated to 0 (discarded). Define the components of $h_{\mu,\nu}$ accordingly

> $Define(\textbf{(44)})$

Defined objects with tensor properties

$$\left\{\gamma_\mu, \sigma_\mu, \partial_\mu, g_{\mu,\nu}, h_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}, X_\mu\right\} \quad \textbf{(45)}$$

Define also a tensor $\eta_{\mu,\nu}$ representing the unperturbed Minkowski metric

> $\mathrm{eta}[\mathrm{mu},\mathrm{nu}] = rhs(\textbf{(43)})$

$$\eta_{\mu,\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \textbf{(46)}$$

> $Define((\mathbf{46}))$

Defined objects with tensor properties

$$\left\{\gamma_{\mu}, \sigma_{\mu}, \partial_{\mu}, \eta_{\mu,\nu}, g_{\mu,\nu}, h_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}, X_{\mu}\right\} \tag{47}$$

The weakly perturbed metric is given by

> g_[mu, nu] = eta[mu, nu] + h[mu, nu]

$$g_{\mu,\nu} = \eta_{\mu,\nu} + h_{\mu,\nu} \tag{48}$$

Make this be the definition of the metric

> $Define((\mathbf{48}))$

Coordinates: $[x, y, z, t]$. *Signature:* $(- - - +)$

$$g_{\mu,\nu} = \begin{bmatrix} -1 + \delta_{1,1}(X) & \delta_{1,2}(X) & \delta_{1,3}(X) & \delta_{1,4}(X) \\ \delta_{1,2}(X) & -1 + \delta_{2,2}(X) & \delta_{2,3}(X) & \delta_{2,4}(X) \\ \delta_{1,3}(X) & \delta_{2,3}(X) & -1 + \delta_{3,3}(X) & \delta_{3,4}(X) \\ \delta_{1,4}(X) & \delta_{2,4}(X) & \delta_{3,4}(X) & 1 + \delta_{4,4}(X) \end{bmatrix}$$

Setting lowercaselatin_is letters to represent space indices

Defined objects with tensor properties

$$\left\{\nabla_{\mu}, \gamma_{\mu}, \sigma_{\mu}, R_{\mu,\nu}, R_{\mu,\nu,\alpha,\beta}, C_{\mu,\nu,\alpha,\beta}, \partial_{\mu}, \eta_{\mu,\nu}, g_{\mu,\nu}, \gamma_{i,j}, h_{\mu,\nu}, \Gamma_{\mu,\nu,\alpha}, G_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}, X_{\mu}\right\} \tag{49}$$

Linearizing the Ricci tensor

The linearized form of the Ricci tensor is computed by introducing this weakly perturbed metric **(48)** in the expression of the Ricci tensor as a function of the metric. This can be accomplished in different ways, the simpler being to use the conversion network between tensors, but for illustration purposes, showing steps one at time, a substitution of definitions one into the other one is used

> $Ricci[definition]$

$$R_{\mu,\nu} = \partial_{\alpha}\left(\Gamma^{\alpha}{}_{\mu,\nu}\right) - \partial_{\nu}\left(\Gamma^{\alpha}{}_{\mu,\alpha}\right) + \Gamma^{\beta}{}_{\mu,\nu}\Gamma^{\alpha}{}_{\beta,\alpha} - \Gamma^{\beta}{}_{\mu,\alpha}\Gamma^{\alpha}{}_{\nu,\beta} \tag{50}$$

> $Christoffel$[~alpha, mu, nu, $definition$]

$$\Gamma^{\alpha}{}_{\mu,\nu} = \frac{g^{\alpha,\beta}\left(\partial_{\nu}\left(g_{\beta,\mu}\right) + \partial_{\mu}\left(g_{\beta,\nu}\right) - \partial_{\beta}\left(g_{\mu,\nu}\right)\right)}{2} \tag{51}$$

> $Substitute((\mathbf{51}), (\mathbf{50}))$

$$R_{\mu,\nu} = \partial_{\alpha}\left(\frac{g^{\alpha,\kappa}\left(\partial_{\nu}(g_{\kappa,\mu}) + \partial_{\mu}(g_{\kappa,\nu}) - \partial_{\kappa}(g_{\mu,\nu})\right)}{2}\right) - \partial_{\nu}\left(\frac{g^{\alpha,\tau}\left(\partial_{\mu}(g_{\tau,\alpha}) + \partial_{\alpha}(g_{\tau,\mu}) - \partial_{\tau}(g_{\alpha,\mu})\right)}{2}\right) + \frac{g^{\beta,\iota}\left(\partial_{\nu}(g_{\iota,\mu}) + \partial_{\mu}(g_{\iota,\nu}) - \partial_{\iota}(g_{\mu,\nu})\right) g^{\alpha,\lambda}\left(\partial_{\beta}(g_{\lambda,\alpha}) + \partial_{\alpha}(g_{\lambda,\beta}) - \partial_{\lambda}(g_{\alpha,\beta})\right)}{4} - \frac{g^{\beta,\omega}\left(\partial_{\mu}(g_{\omega,\alpha}) + \partial_{\alpha}(g_{\omega,\mu}) - \partial_{\omega}(g_{\alpha,\mu})\right) g^{\alpha,\chi}\left(\partial_{\nu}(g_{\chi,\beta}) + \partial_{\beta}(g_{\chi,\nu}) - \partial_{\chi}(g_{\beta,\nu})\right)}{4} \tag{52}$$

Introducing $(\mathbf{48}) \equiv g_{\mu,\nu} = \eta_{\mu,\nu} + h_{\mu,\nu}$, and also the inert form of the Ricci tensor to facilitate simplification some steps below,

> $Substitute((\mathbf{48}), Ricci = \%Ricci, (\mathbf{52}))$

$$R_{\mu,\nu} = \frac{\partial_{\alpha}\left(\eta^{\alpha,\kappa} + h^{\alpha,\kappa}\right)\left(\partial_{\nu}\left(\eta_{\kappa,\mu} + h_{\kappa,\mu}\right) + \partial_{\mu}\left(\eta_{\kappa,\nu} + h_{\kappa,\nu}\right) - \partial_{\kappa}\left(\eta_{\mu,\nu} + h_{\mu,\nu}\right)\right)}{2} + \frac{1}{2}\left(\left(\eta^{\alpha,\kappa} + h^{\alpha,\kappa}\right)\left(\partial_{\alpha}\left(\partial_{\nu}\left(\eta_{\kappa,\mu} + h_{\kappa,\mu}\right)\right) + \partial_{\alpha}\left(\partial_{\mu}\left(\eta_{\kappa,\nu} + h_{\kappa,\nu}\right)\right) - \partial_{\alpha}\left(\partial_{\kappa}\left(\eta_{\mu,\nu} + h_{\mu,\nu}\right)\right)\right)\right) - \frac{\partial_{\nu}\left(\eta^{\alpha,\tau} + h^{\alpha,\tau}\right)\left(\partial_{\mu}\left(\eta_{\alpha,\tau} + h_{\alpha,\tau}\right) + \partial_{\alpha}\left(\eta_{\mu,\tau} + h_{\mu,\tau}\right) - \partial_{\tau}\left(\eta_{\alpha,\mu} + h_{\alpha,\mu}\right)\right)}{2} - \frac{1}{2}\left(\left(\eta^{\alpha,\tau} + h^{\alpha,\tau}\right)\left(\partial_{\mu}\left(\partial_{\nu}\left(\eta_{\alpha,\tau} + h_{\alpha,\tau}\right)\right) + \partial_{\alpha}\left(\partial_{\nu}\left(\eta_{\mu,\tau} + h_{\mu,\tau}\right)\right) - \partial_{\nu}\left(\partial_{\tau}\left(\eta_{\alpha,\mu} + h_{\alpha,\mu}\right)\right)\right)\right) + \frac{1}{4}\left(\left(\eta^{\beta,\iota} + h^{\beta,\iota}\right)\left(\partial_{\nu}\left(\eta_{\iota,\mu} + h_{\iota,\mu}\right) + \partial_{\mu}\left(\eta_{\iota,\nu} + h_{\iota,\nu}\right) - \partial_{\iota}\left(\eta_{\mu,\nu} + h_{\mu,\nu}\right)\right)\left(\eta^{\alpha,\lambda} + h^{\alpha,\lambda}\right)\left(\partial_{\beta}\left(\eta_{\alpha,\lambda} + h_{\alpha,\lambda}\right) + \partial_{\alpha}\left(\eta_{\beta,\lambda} + h_{\beta,\lambda}\right) - \partial_{\lambda}\left(\eta_{\alpha,\beta} + h_{\alpha,\beta}\right)\right)\right) - \frac{1}{4}\left(\left(\eta^{\beta,\omega} + h^{\beta,\omega}\right)\left(\partial_{\mu}\left(\eta_{\alpha,\omega} + h_{\alpha,\omega}\right) + \partial_{\alpha}\left(\eta_{\mu,\omega} + h_{\mu,\omega}\right) - \partial_{\omega}\left(\eta_{\alpha,\mu} + h_{\alpha,\mu}\right)\right)\left(\eta^{\alpha,\chi} + h^{\alpha,\chi}\right)\left(\partial_{\nu}\left(\eta_{\beta,\chi} + h_{\beta,\chi}\right) + \partial_{\beta}\left(\eta_{\chi,\nu} + h_{\chi,\nu}\right) - \partial_{\chi}\left(\eta_{\beta,\nu} + h_{\beta,\nu}\right)\right)\right) \tag{53}$$

This expression contains several terms quadratic in the small perturbation $h_{\mu,\nu}$ and its derivatives. The **new in Maple 2025** routine to filter out those terms is Physics:-Library:-Linearize, which requires specifying the symbol representing the small quantities

> $Library{:-}Linearize((\mathbf{53}), h)$

$$R_{\mu,\nu} = \frac{\eta^{\alpha,\tau}\,\partial_{\nu}(\partial_{\tau}(h_{\alpha,\mu}))}{2} - \frac{\eta^{\alpha,\tau}\,\partial_{\mu}(\partial_{\nu}(h_{\alpha,\tau}))}{2} - \frac{\eta^{\alpha,\kappa}\,\partial_{\alpha}(\partial_{\kappa}(h_{\mu,\nu}))}{2} + \frac{\eta^{\alpha,\kappa}\,\partial_{\alpha}(\partial_{\nu}(h_{\kappa,\mu}))}{2} + \frac{\eta^{\alpha,\kappa}\,\partial_{\alpha}(\partial_{\mu}(h_{\kappa,\nu}))}{2} - \frac{\eta^{\alpha,\tau}\,\partial_{\alpha}(\partial_{\nu}(h_{\mu,\tau}))}{2} \quad \mathbf{(54)}$$

Important: in this result, $\eta_{\mu,\nu}$ is the flat Minkowski metric, not the perturbed metric $g_{\mu,\nu}$. However, in the *context of a linearized formulation*, $\eta_{\mu,\nu}$ raises and lowers tensor indices the same way as $g_{\mu,\nu}$. Hence, to further simplify contracted products of $\eta_{\mu,\nu}$ in **(54)**, it is practical to reintroduce $g_{\mu,\nu}$ representing that Minkowski metric and simplify using the internal algorithms for a flat metric

> $g\_[\min]$

The Minkowski metric in coordinates $[x, y, z, t]$

Signature: $(- - - +)$

$$g_{\mu,\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{(55)}$$

To proceed simplifying, replace in the expression **(54)** for the Ricci tensor the intermediate Minkowski $\eta_{\mu,\nu}$ by $g_{\mu,\nu}$

> $subs(\mathrm{eta} = g\_, (\mathbf{54}))$

$$R_{\mu,\nu} = \frac{g^{\alpha,\tau}\,\partial_{\nu}(\partial_{\tau}(h_{\alpha,\mu}))}{2} - \frac{g^{\alpha,\tau}\,\partial_{\mu}(\partial_{\nu}(h_{\alpha,\tau}))}{2} - \frac{g^{\alpha,\kappa}\,\partial_{\alpha}(\partial_{\kappa}(h_{\mu,\nu}))}{2} + \frac{g^{\alpha,\kappa}\,\partial_{\alpha}(\partial_{\nu}(h_{\kappa,\mu}))}{2} + \frac{g^{\alpha,\kappa}\,\partial_{\alpha}(\partial_{\mu}(h_{\kappa,\nu}))}{2} - \frac{g^{\alpha,\tau}\,\partial_{\alpha}(\partial_{\nu}(h_{\mu,\tau}))}{2} \quad \mathbf{(56)}$$

Simplifying, results in the linearized form of the Ricci tensor shown in the Wikipedia page for Linearized gravity.

> $Simplify((\mathbf{56}))$

$$R_{\mu,\nu} = -\frac{\partial_{\mu}(\partial_{\nu}(h_{\tau}^{\ \tau}))}{2} - \frac{\Box(h_{\mu,\nu})}{2} + \frac{\partial_{\nu}(\partial_{\tau}(h_{\mu}^{\ \tau}))}{2} + \frac{\partial_{\mu}(\partial_{\tau}(h_{\nu}^{\ \tau}))}{2} \quad \mathbf{(57)}$$

Linearizing Einstein's equations

Einstein's equations are the components of Einstein's tensor, whose definition in terms of the Ricci tensor is

> *Einstein*[*definition*]

$$G_{\mu,\nu} = R_{\mu,\nu} - \frac{g_{\mu,\nu} R_{\alpha}^{\ \alpha}}{2} \tag{58}$$

Compute the trace $R_{\alpha}^{\ \alpha}$ directly from the linearized form **(57)** of the Ricci tensor,

> $g\_[\mathrm{mu}, \mathrm{nu}] \cdot$ **(57)**

$$R_{\mu,\nu}\, g^{\mu,\nu} = \left(-\frac{\partial_{\mu}\left(\partial_{\nu}\left(h_{\tau}^{\ \tau}\right)\right)}{2} - \frac{\square\left(h_{\mu,\nu}\right)}{2} + \frac{\partial_{\nu}\left(\partial_{\tau}\left(h_{\mu}^{\ \tau}\right)\right)}{2} + \frac{\partial_{\mu}\left(\partial_{\tau}\left(h_{\nu}^{\ \tau}\right)\right)}{2} \right) g^{\mu,\nu} \tag{59}$$

> *Simplify*(**(59)**)

$$R_{\nu}^{\ \nu} = -\square\left(h_{\alpha}^{\ \alpha}\right) + \partial_{\alpha}\left(\partial_{\tau}\left(h^{\alpha,\tau}\right)\right) \tag{60}$$

The linearized Einstein equations are constructed reproducing the definition **(58)** using **(57)** and **(60)**

> **(57)** $- \frac{1}{2} g\_[\mathrm{mu}, \mathrm{nu}] \cdot$ **(60)**

$$R_{\mu,\nu} - \frac{g_{\mu,\nu} R_{\alpha}^{\ \alpha}}{2} = -\frac{\partial_{\mu}\left(\partial_{\nu}\left(h_{\tau}^{\ \tau}\right)\right)}{2} - \frac{\square\left(h_{\mu,\nu}\right)}{2} + \frac{\partial_{\nu}\left(\partial_{\tau}\left(h_{\mu}^{\ \tau}\right)\right)}{2} + \frac{\partial_{\mu}\left(\partial_{\tau}\left(h_{\nu}^{\ \tau}\right)\right)}{2} - \frac{g_{\mu,\nu}\left(-\square\left(h_{\alpha}^{\ \alpha}\right) + \partial_{\alpha}\left(\partial_{\tau}\left(h^{\alpha,\tau}\right)\right)\right)}{2} \tag{61}$$

which is the same formula shown in the Wikipedia page for Linearized gravity.

You can now redefine the general $h_{\mu,\nu}$ introduced in **(44)** in different ways (see discussion in the Wikipedia page), or, depending on the case, just *substitute* your preferred gauge in this formula **(61)** for the general case. For example, the condition for the *Harmonic gauge* also known as *Lorentz gauge* reduces the linearized field equations to their simplest form

> $d\_[\mathrm{mu}](h[\sim\mathrm{mu}, \mathrm{nu}]) = \frac{1}{2} d\_[\mathrm{nu}](h[\mathrm{alpha}, \mathrm{alpha}])$

$$\partial_{\mu}\left(h^{\mu}_{\ \nu}\right) = \frac{\partial_{\nu}\left(h_{\alpha}^{\ \alpha}\right)}{2} \tag{62}$$

> *Substitute*(**(62)**, **(61)**)

(63)

$$R_{\mu,\nu} - \frac{g_{\mu,\nu} R_{\alpha}^{\ \alpha}}{2} = -\frac{\partial_{\mu}\left(\partial_{\nu}\left(h_{\tau}^{\ \tau}\right)\right)}{2} - \frac{\Box\left(h_{\mu,\nu}\right)}{2} + \frac{\partial_{\nu}\left(\frac{\partial_{\mu}\left(h_{\lambda}^{\ \lambda}\right)}{2}\right)}{2} + \frac{\partial_{\mu}\left(\frac{\partial_{\nu}\left(h_{\kappa}^{\ \kappa}\right)}{2}\right)}{2} - \frac{g_{\mu,\nu}\left(-\Box\left(h_{\alpha}^{\ \alpha}\right) + \partial_{\alpha}\left(\frac{\partial^{\alpha}\left(h_{\beta}^{\ \beta}\right)}{2}\right)\right)}{2} \tag{63}$$

> *Simplify*((**63**))

$$R_{\mu,\nu} - \frac{g_{\mu,\nu} R_{\alpha}^{\ \alpha}}{2} = -\frac{\Box\left(h_{\mu,\nu}\right)}{2} + \frac{\Box\left(h_{\alpha}^{\ \alpha}\right) g_{\mu,\nu}}{4} \tag{64}$$

Relative Tensors

In General Relativity, the context of a curved spacetime, it is sometimes necessary to work with *relative* tensors, for which the transformation rule under a transformation of coordinates involves powers of the determinant of the transformation - see Chapter 4 of "Lovelock, D., and Rund, H. **Tensors, Differential Forms and Variational Principles**, Dover, 1989." Physics in Maple 2025 includes a complete, new implementation of relative tensors.

To indicate that a tensor being defined is *relative* pass its *relative weight*. For example, set a curved spacetime,

```
> restart;
  with(Physics) :
  g_[sc];
```

Systems of spacetime coordinates are: $\{X = (r, \theta, \phi, t)\}$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (r, \theta, \phi, t)\}$

Setting lowercaselatin_is letters to represent space indices

The Schwarzschild metric in coordinates $[r, \theta, \phi, t]$

Parameters: $[m]$

Signature: $(- - - +)$

---

(65)

$$g_{\mu,\nu} = \begin{bmatrix} \frac{r}{2m-r} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & \frac{r-2m}{r} \end{bmatrix} \tag{65}$$

Define now two tensors of one index, one of them being *relative*

> $Define(T[\mu])$

Defined objects with tensor properties

$$\{\nabla_\mu, \gamma_\mu, \sigma_\mu, R_{\mu,\nu}, R_{\mu,\nu,\alpha,\beta}, T_\mu, C_{\mu,\nu,\alpha,\beta}, \partial_\mu, g_{\mu,\nu}, \gamma_{i,j}, \Gamma_{\mu,\nu,\alpha}, G_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}, X_\mu\} \tag{66}$$

> $Define(R[\mu], relativeweight = 1)$

Defined objects with tensor properties

$$\{\nabla_\mu, \gamma_\mu, \sigma_\mu, R_\mu, R_{\mu,\nu}, R_{\mu,\nu,\alpha,\beta}, T_\mu, C_{\mu,\nu,\alpha,\beta}, \partial_\mu, g_{\mu,\nu}, \gamma_{i,j}, \Gamma_{\mu,\nu,\alpha}, G_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}, X_\mu\} \tag{67}$$

Transformation of Coordinates

Consider a transformation of coordinates, from spherical (r, θ, ϕ, t) to (ρ, θ, ϕ, t) where

> $TR := r = \left(1 + \frac{m}{2\rho}\right)^2 \rho$

$$TR := r = \left(1 + \frac{m}{2\rho}\right)^2 \rho \tag{68}$$

The transformed components of T_μ and R_μ are, respectively,

> $TransformCoordinates(TR, T[\mu], [\rho, \theta, \phi, t])$

$$\begin{bmatrix} -\frac{(m^2 - 4\rho^2)\, T_1}{4\rho^2} \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} \tag{69}$$

> $TransformCoordinates(TR, R[\mu], [\rho, \theta, \phi, t])$

$$\begin{bmatrix} \frac{(m^2 - 4\rho^2)^2 R_1}{16\rho^4} \\ -\frac{(m^2 - 4\rho^2) R_2}{4\rho^2} \\ -\frac{(m^2 - 4\rho^2) R_3}{4\rho^2} \\ -\frac{(m^2 - 4\rho^2) R_4}{4\rho^2} \end{bmatrix} \tag{70}$$

where, when comparing both results, we see that the transformed components for R_μ are all multiplied by J^n with $n = 1$ and J is the determinant of the transformation:

> $J_{matrix} := simplify(VectorCalculus:\text{-}Jacobian([rhs(TR), \theta, \phi, t], [\rho, \theta, \phi, t]))$

$$J_{matrix} := \begin{bmatrix} \frac{-m^2 + 4\rho^2}{4\rho^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{71}$$

> $J = LinearAlgebra:\text{-}Determinant(J_{matrix})$

$$J = \frac{-m^2 + 4\rho^2}{4\rho^2} \tag{72}$$

Relative weight

The relative weight of a scalar, tensor or tensorial expression can be computed using the Physics:-Library:-GetRelativeWeight command. For the two tensors T_μ and R_μ used above,

> $Library:\text{-}GetRelativeWeight(T[\mathrm{mu}])$

$$0 \tag{73}$$

> $Library:\text{-}GetRelativeWeight(R[\mathrm{mu}])$

$$1 \tag{74}$$

The relative weight of a tensor does not depend on the covariant or contravariant character of its indices

> $Library:\text{-}GetRelativeWeight(R[\sim\mathrm{mu}])$

$$1 \tag{75}$$

The LeviCivita tensor is a special case, has its relative weight defined when Physics is loaded, and because in a curved spacetime it is not a tensor its relative weight depends on the covariant or contravariant character of its indices

> *Library*:-*GetRelativeWeight*(*LeviCivita*[alpha, beta, mu, nu])

$$-1 \tag{76}$$

> *Library*:-*GetRelativeWeight*(*LeviCivita*[~alpha, ~beta, ~mu,~nu])

$$1 \tag{77}$$

The relative weight *w* of a product is equal to the sum of relative weights of each factor

> $R[\mathrm{mu}]^2$

$$R_{\mu} R^{\mu} \tag{78}$$

> *Library*:-*GetRelativeWeight*(**(78)**)

$$2 \tag{79}$$

The relative weight *w* of a power is equal to the relative weight of the base multiplied by the power

> $\dfrac{1}{R[\mathrm{mu}]^2}$

$$\frac{1}{R_{\mu} R^{\mu}} \tag{80}$$

Library:-*GetRelativeWeight*(**(80)**)

$$-2 \tag{81}$$

The relative weight *w* of a sum is equal to the relative weight of one of its terms and exists if all the terms have the same *w*.

> *R*[~mu] + *LeviCivita*[~alpha, ~beta, ~mu,~nu]·*T*[alpha] *T*[beta] *T*[nu]

$$\epsilon^{\alpha, \beta, \mu, \nu} T_{\alpha} T_{\beta} T_{\nu} + R^{\mu} \tag{82}$$

> *Library*:-*GetRelativeWeight*(**(82)**)

$$1 \tag{83}$$

The relative weight of any *determinant* is always equal to 2

> *%g_*[*determinant*]

$$|\mathrm{g}| \tag{84}$$

> *Library*:-*GetRelativeWeight*(**(84)**)

$$2 \tag{85}$$

Relative Term in covariant derivatives

When computing the covariant derivative of a relative scalar, tensor or tensorial expression that has non-zero relative weight *w*, a relative term is added, that can be computed using the Physics:-Library:-

GetRelativeWeight command.

> $g_{det} := \%g\_[:\text{-}determinant];$

$$g_{det} := |\,g\,| \qquad \textbf{(86)}$$

> $Library\text{:-}GetRelativeTerm\left(g_{det}, \mathrm{mu}\right);$

$$-2\,\Gamma^{\nu}{}_{\mu,\,\nu}\,|\,g\,| \qquad \textbf{(87)}$$

Consequently,

> $(\%D\_[\mathrm{mu}] = D\_[\mathrm{mu}])\,(g_{det});$

$$\nabla_{\mu}(|\,g\,|) = 0 \qquad \textbf{(88)}$$

To understand this zero value on the right-hand side, express the left-hand side in terms of d_

> $convert(\textbf{(88)}, \boldsymbol{d\_})$

$$\partial_{\mu}(|\,g\,|) - 2\,\Gamma^{\alpha}{}_{\alpha,\,\mu}\,|\,g\,| = 0 \qquad \textbf{(89)}$$

evaluate the inert $\%d\_$

> $factor(eval(\textbf{(89)}, \%d\_ = d\_))$

$$|\,g\,|\left(g^{\alpha,\,\nu}\,\partial_{\mu}\left(g_{\alpha,\,\nu}\right) - 2\,\Gamma^{\alpha}{}_{\alpha,\,\mu}\right) = 0 \qquad \textbf{(90)}$$

The factor in parentheses is equal to $g^{\alpha,\,\nu}\,\nabla_{\mu}\left(g_{\alpha,\,\nu}\right)$, where the covariant derivative of the metric is equal to zero, so

> $Simplify(\textbf{(90)})$

$$0 = 0 \qquad \textbf{(91)}$$

Consider the covariant derivative of T_{μ} and R_{μ} defined in **(66)** and **(67)**

> $Library\text{:-}GetRelativeWeight(T[\mathrm{mu}])$

$$0 \qquad \textbf{(92)}$$

> $Library\text{:-}GetRelativeWeight(R[\mathrm{mu}])$

$$1 \qquad \textbf{(93)}$$

The corresponding covariant derivatives

> $(\%D\_[\mathrm{mu}] = D\_[\mathrm{mu}])\,(T[\mathrm{mu}](X));$

$$\nabla_{\mu}\left(T^{\mu}(X)\right) = \nabla_{\mu}\left(T^{\mu}(X)\right) \qquad \textbf{(94)}$$

> $expand(\textbf{(94)})$

$$\nabla_{\mu}\left(T^{\mu}(X)\right) = \frac{2\,T^{\mu}(X)\,\partial_{\mu}(r)}{r} + \frac{\partial_{\mu}(\theta)\,\cos(\theta)\,T^{\mu}(X)}{\sin(\theta)} + \partial_{\mu}\left(T^{\mu}(X)\right) \qquad \textbf{(95)}$$

> $(\%D\_[\mathrm{mu}] = D\_[\mathrm{mu}])\,(R[\mathrm{mu}](X));$

(96)

$$\nabla_{\mu}\left(R^{\mu}(X)\right) = \nabla_{\mu}\left(R^{\mu}(X)\right) \tag{96}$$

> *expand*((96))

$$\nabla_{\mu}\left(R^{\mu}(X)\right) = \frac{2\, R^{\mu}(X)\, \partial_{\mu}(r)}{r} + \frac{\partial_{\mu}(\theta)\cos(\theta)\, R^{\mu}(X)}{\sin(\theta)} + \partial_{\mu}\left(R^{\mu}(X)\right) - \Gamma^{\nu}{}_{\mu,\nu} R^{\mu}(X) \tag{97}$$

where in the above we see the additional (relative) term

> *Library*:-*GetRelativeTerm*(*R*[~mu](*X*), mu)

$$-\Gamma^{\nu}{}_{\mu,\nu} R^{\mu}(X) \tag{98}$$

New Physics:-Library commands

ConvertToF, *Linearize*, GetRelativeTerm, GetRelativeWeight.

Examples

- ***ConvertToF*** receives an algebraic expression involving tensors and/or tensor functions and rewrites them in terms of the tensor of name F when that is possible. This routine is similar, however more general than the standard convert which only handles the existing conversion network for the tensors of General Relativity in that ConvertToF also uses any tensor definition you introduce using Define, expressing a tensor in terms of others.

  Load any curved spacetime metric automatically setting the coordinates

restart;
with(*Physics*) :
*g_*[*sc*];

Systems of spacetime coordinates are: $\{X = (r, \theta, \phi, t)\}$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (r, \theta, \phi, t)\}$

Setting lowercaselatin_is letters to represent space indices

The Schwarzschild metric in coordinates $[r, \theta, \phi, t]$

Parameters: $[m]$

Signature: $(- - - +)$

$$g_{\mu,\nu} = \begin{bmatrix} \frac{r}{2\,m - r} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & \frac{r - 2\,m}{r} \end{bmatrix} \quad \mathbf{(99)}$$

For example, rewrite the Christoffel symbols in terms of the metric g_; this works as in previous releases

> *Christoffel*[μ, α, β] = *Library:-ConvertToF*(*Christoffel*[μ, α, β], *g_*);

$$\Gamma_{\mu,\alpha,\beta} = \frac{\partial_\beta(g_{\alpha,\mu})}{2} + \frac{\partial_\alpha(g_{\beta,\mu})}{2} - \frac{\partial_\mu(g_{\alpha,\beta})}{2} \quad \mathbf{(100)}$$

Define a A_μ representing the 4D electromagnetic potential as a function of the coordinates X and $F_{\mu,\nu}$ representing the electromagnetic field tensors

> *Define*(*A*[μ] = *A*[μ](*X*), *quiet*);

$$\left\{A_\mu, \nabla_\mu, \gamma_\mu, \sigma_\mu, R_{\mu,\nu}, R_{\mu,\nu,\alpha,\beta}, C_{\mu,\nu,\alpha,\beta}, \partial_\mu, g_{\mu,\nu}, \gamma_{i,j}, \Gamma_{\mu,\nu,\alpha}, G_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}, X_\mu\right\} \quad \mathbf{(101)}$$

> *Define*(*F*[μ, ν] = *d_*[μ](*A*[ν]) − *d_*[ν](*A*[μ]));

Defined objects with tensor properties

$$\left\{A_\mu, \nabla_\mu, \gamma_\mu, F_{\mu,\nu}, \sigma_\mu, R_{\mu,\nu}, R_{\mu,\nu,\alpha,\beta}, C_{\mu,\nu,\alpha,\beta}, \partial_\mu, g_{\mu,\nu}, \gamma_{i,j}, \Gamma_{\mu,\nu,\alpha}, G_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}, X_\mu\right\} \quad \mathbf{(102)}$$

Rewrite the following expression in terms of the electromagnetic potential A_μ

> *F*[μ, ν] = *Library:-ConvertToF*(*F*[μ, ν], *A*);

$$F_{\mu,\nu} = \partial_\mu(A_\nu) - \partial_\nu(A_\mu) \quad \mathbf{(103)}$$

In the example above, the output is similar to this other one

> *F*[*definition*];

$$F_{\mu,\nu} = \partial_\mu(A_\nu) - \partial_\nu(A_\mu) \quad \mathbf{(104)}$$

The rewriting, however, works also with tensorial expressions

> *F*[mu, nu] * *A*[mu] * *A*[nu]

$$F_{\mu,\nu} A^\mu A^\nu \quad \mathbf{(105)}$$

> *Library:-ConvertToF*(**(105)**, *A*);

$$\left(\partial_\mu(A_\nu) - \partial_\nu(A_\mu)\right) A^\mu A^\nu \quad \mathbf{(106)}$$

- ***Linearize*** receives a tensorial expression T and an indication of the small quantities h in T , and discards terms quadratic or of higher order in h. For an example of this new routine in action, see the section Linearized Gravity above.

- ***GetRelativeTerm*** and ***GetRelativeWeight*** are illustrated in the section Relative Tensors above.